

## CONSISTENT DISCRETE GRAVITY SOLUTION OF THE PROBLEM OF TIME: A MODEL

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The recently introduced consistent discrete lattice formulation of canonical general relativity produces a discrete theory that is constraint-free. This immediately allows to overcome several of the traditional obstacles posed by the “problem of time” in totally constrained systems and quantum gravity and cosmology. In particular, one can implement the Page–Wootters relational quantization. This brief paper discusses this idea in the context of a simple model system –the parameterized particle– that is usually considered one of the crucial tests for any proposal for solution to the problem of time in quantum gravity.

### 1. Introduction

There have always been problems to formulate general relativity on a lattice. Even classically, if one simply discretizes Einstein’s equations, the resulting discrete equations are an inconsistent set of algebraic equations, that is, they cannot be solved simultaneously. This is well known in numerical relativity, where if one starts with a solution of the discrete constraint equations initially, upon evolution with the discrete evolution equations the resulting metric and extrinsic curvature will generically fail to solve the constraints.

A solution to this problem has recently been proposed<sup>1,2</sup>, following the early work of T.D. Lee<sup>3</sup> in simple mechanical systems. It consists in choosing the Lagrange multipliers (in the case of the usual canonical formulations of general relativity the lapse and the shift) so that the evolved quantities solve the constraints. One has four equations for four unknowns so this

is in principle possible. This proposal has been explored in some detail in the context of cosmologies<sup>4</sup>, where the resulting equations for the Lagrange multipliers are simple to solve. A canonical formulation for these formulations has also been introduced<sup>1</sup>.

One of the attractive features of the resulting canonical theories is that the constraints are gone, since they are solved for the Lagrange multipliers. The presence of the constraints is one of the major roadblocks in the quantization of gravity. In particular most aspects of the “problem of time” are related to the presence of the constraints. The lack of constraints in the lattice formulation should therefore allow to make considerable inroads into these problems. We will discuss this in general in a separate publication. In this short paper we would like to exhibit how the construction works, in detail, in a model system that many people consider to embody the germ of many of the difficulties of the “problem of time” in quantum gravity: the parameterized particle.

In particular we would like to exhibit how the definition of a relational notion of time, as proposed in detail by Page and Wootters<sup>5</sup> (but with a long conceptual history going back to Leibniz, Mach, Einstein, DeWitt, Hartle, Unruh, Barbour, Rovelli, Crane, Smolin and many others) operates in detail in our proposal.

This paper is organized as follows. We start with a brief recap of the proposal of Page and Wootters and the criticism it faced (eloquently articulated by Kuchař<sup>6</sup>). We then briefly summarize in section 3 our consistent canonical formulation of discrete mechanical systems. In the fourth section we apply the construction to the parameterized particle. We end with a summary and proposal for further implications of this formalism.

## 2. The Page–Wootters construction

The usual textbook formulation of quantum mechanics presupposes the existence of an externally defined, classical quantity called “ $t$ ”. The wavefunctions of the Schrödinger representation are functions  $\Psi(x, t)$ . The variable  $x$  has a very different role than  $t$ , it is represented by an operator of which one can —for instance— compute expectation values. The time variable  $t$ , on the other hand, remains always classical. This is clearly only an approximate concept that requires the existence of a classical clock external to the quantum system. It is therefore not a very useful notion in the context of closed systems where everything behaves quantum mechanically, as for instance cosmology close to the Big Bang.

Page and Wootters<sup>5</sup> proposed an alternative to this. They consider a system with variables  $q_1, q_2, \dots$ . One then considers wavefunctions  $\Psi(q_1, q_2, \dots)$  where all variables are now quantum mechanical in nature and are represented by operators. One then chooses one of the variables and calls it “time”, for instance  $q_1 = T$ . Being a quantum variable, one now considers the calculation of conditional probabilities in the sense of “what is the probability that the variable  $q_k$  takes a value in a certain range  $\Delta q_k$  near  $q_k^0$  when “time” takes a value in a range  $\Delta T$  near  $T^0$ .” This is how the Page–Wootters quantum mechanics is constructed.

This egalitarian framework appears very attractive in the context of quantum cosmology. However, there is a glitch. General relativity is a constrained theory. In principle, one should consider classical variables  $q_1, \dots$ , etc. that are “observables” (that is, have vanishing Poisson brackets with the constraints). Unfortunately since one of the constraints is the generator of evolution, it means these variables are “perennials” (as Kuchař usually calls them). This implies that one cannot expect any of them to work as a clock. The resulting quantum mechanics will have no evolution. Page and Wootters tried to circumvent this by considering  $q_1, \dots$  to be “kinematical” variables (for instance the  $g_{12}$  component of the metric in some coordinates) that do not have vanishing Poisson brackets with the constraints. In terms of these variables one would expect to see “evolution”. But here is where the presence of the constraints generates problems. Kinematical variables do not have a well defined action as quantum operators on states that are annihilated by the constraints. Worse, such states are expected to have a distributional nature as a subset of the full space of states. This implies that they do not really admit a probabilistic interpretation (see Kuchař<sup>6</sup> for more details). This has a concrete consequence. When one uses the conditional probabilities of Page and Wootters to compute a propagator, one gets the wrong result that the system does not propagate (this can be seen in detail in the example of the parameterized particle we discuss later<sup>6</sup>).

Because of these problems and in spite of its initial naturalness and appeal, the Page–Wootters proposal has been considered to fall short of presenting a solution to the problem of time in quantum gravity. As is made clear in the above discussion, it is the presence of the Hamiltonian constraint in general relativity what really complicates the application of this proposal. In the consistent discrete canonical formulation of general relativity the constraints are not present. Therefore it opens the possibility to revisit the Page and Wootters proposal.

### 3. Consistent discrete mechanics

Since in this paper we will not really discuss general relativity but only the example of the parameterized particle, it suffices to discuss the consistent discrete approach in the context of mechanical systems (in field theories additional subtleties appear, although the general construction is similar<sup>1</sup>).

We start by considering a system with an action as in ordinary classical mechanics  $S = \int dt L(q, \dot{q})$ . We then discretize time, substituting the time derivative by  $\dot{q} = (q_{n+1} - q_n)/\Delta t$  and the integral in the action becomes a sum. We write the action as  $S = \sum_{i=1}^n L(q_i, q_{i+1})$ . One can work out the Lagrange equations. Then one would usually introduce a Hamiltonian via a Legendre transform. We would like to argue that this is not the most natural thing to do in this context. After all, a Hamiltonian is an infinitesimal generator of time evolution, and in a context where time is discrete there is no natural meaning to an infinitesimal time evolution generator. What is natural is to introduce a finite generator. This is accomplished by a canonical transformation that takes the system from  $q_n$  to  $q_{n+1}$ . A generating function  $F$  of a type I canonical transformation (i.e. mapping from  $q, p$  to  $Q, P$  via  $P = -\partial F(q, Q)/\partial Q$ ,  $p = \partial F(q, Q)/\partial q$ ), that implements the time evolution is simply given by  $F(q_i, q_{i+1}) = -L(q_i, q_{i+1}) \equiv -L(i, i+1)$ . The evolution equations are therefore,

$$p_{n+1} = \frac{\partial L(q_n, q_{n+1})}{\partial q_{n+1}}, \quad p_n = -\frac{\partial L(q_n, q_{n+1})}{\partial q_n}. \quad (1)$$

These equations define the canonical momenta  $p$  and the evolution for the canonical pair. The evolution preserves the Poisson bracket of  $q_i, p_i$ . If we are a bit more explicit, writing out the usual form of the Lagrangian for a constrained system,

$$L(n, n+1) = p_n(q_{n+1} - q_n) - H(q_n, p_n) - \lambda_{nB} \phi^B(q_n, p_n) \quad (2)$$

where we assume we have  $M$  constraints  $\phi^B$  with  $B = 1 \dots M$  and  $\lambda_{nB}$  are Lagrange multipliers. (We are slightly abusing the notation; in order to have  $M$  constraints one needs more than two canonical variables, so one should assume that at each  $n$  one has a multicomponent vector for the  $q$ 's and  $p$ 's). If we work out the equations of the canonical transformation we have,

$$p_n - p_{n-1} = -\frac{\partial H(q_n, p_n)}{\partial q_n} - \lambda_{nB} \frac{\partial \phi^B(q_n, p_n)}{\partial q_n} \quad (3)$$

$$q_{n+1} - q_n = \frac{\partial H(q_n, p_n)}{\partial p_n} + \lambda_{nB} \frac{\partial \phi^B(q_n, p_n)}{\partial p_n} \quad (4)$$

$$\phi^B(q_n, p_n) = 0. \quad (5)$$

These equations look superficially like a straight discretization of the usual Hamilton equations for constrained systems. But there is a significant difference with the continuum case. The last equation (the constraints) cannot hold simultaneously with the two first ones unless one chooses specific ( $p$  and  $q$ -dependent) values for the Lagrange multipliers. With such a choice one is left with a free evolution given by the first two equations with specific values of  $\lambda$  and the constraints are automatically satisfied.

Quantization of the system can be achieved by implementing the canonical transformation as a unitary transformation quantum mechanically. For instance, if one works in the Heisenberg picture, such a transformation should implement equations (3,4) as operator equations. In the Schrödinger picture, wavefunctions are function of the discrete parameter  $n$ .

It is deceiving to think that the above proposal for quantization will end up being useful for systems like general relativity. Although one will construct a quantum theory without conceptual obstructions (the only real obstruction could be computational, i.e. solving the equations for the Lagrange multipliers and then implementing the complicated resulting classical evolution equations operatorially via a unitary transformation) the resulting theory would be difficult to interpret. The discrete parameter  $n$  has no direct physical meaning, it is just a computational entity introduced in the discretization. Therefore it cannot be expected to be correlated with any physically meaningful “time” in a generic situation.

In spite of these conceptual flaws, it should be emphasized that the resulting quantum theory is well defined, with the wavefunctions having a correct probabilistic interpretation. In this we see the first direct positive consequence of not having constraints. The probabilistic interpretation of the wavefunctions, which was problematic in the Hartle–Hawking formulation (see Kuchař<sup>6</sup> for a detailed criticism) is suddenly not a problem here.

The fact that the most straightforward quantization does not appear to be of physical interest, yet yields a good probabilistic interpretation for the wavefunction is what makes the Page–Wootters approach quite attractive. We have at hand the tools needed to compute the conditional probabilities yet we will end up with a theory where the notion of time is introduced physically and the parameter  $n$  only plays a computationally auxiliary role. Another way of putting it is that the discretization technique naturally yields a concrete implementation of the “mysterious time” concept introduced by Unruh<sup>7</sup>.

Conceptually the proposal is therefore clear. The main point of this paper is to show that the proposal actually works in detail in a simple example that has however been quite problematic to tackle.

#### 4. The Page–Wootters quantization of the parameterized particle

We cannot treat the free parameterized particle with our technique. This is due to the fact that its Lagrangian (at least written in the usual form) is simple enough that the discrete evolution equations are automatically consistent without applying our technique. One could perform a change of variables to non-standard variables in which the evolution equations could only be made consistent by determining the Lagrange multipliers, but we will not pursue this route.

We prefer to consider a slightly more general (and in the end more physically realistic) example: a non-relativistic particle in a constant force field (for instance, a charged particle in a constant electric field). The continuum Lagrangian can be written as,

$$S = \int \left[ p\dot{q} + p_0\dot{q}^0 - N\left(p_0 + \frac{p^2}{2m} + \alpha q\right) \right] d\tau, \quad (6)$$

where the dot means derivative with respect to the parameter  $\tau$ ,  $\alpha$  is the constant force on the particle due to the external field,  $N$  is the Lagrange multiplier.  $q, p$  are the usual spatial coordinate and momentum and  $q^0, p_0$  are the temporal coordinate and its conjugate momentum. The problem can be de-parameterized by choosing  $q^0 = \tau$  and one recovers the ordinary mechanics of a particle in a constant force field. If one chooses however  $q = \tau$ , the resulting quantum theory is not consistent, yielding non-self-adjoint operators<sup>8</sup>.

We proceed to discretize the Lagrangian as we outlined in section 3,

$$L(n, n+1) = p^n(q_{n+1} - q_n) + p_0^n(q_{n+1}^0 - q_n^0) - N_n\left(p_0^n + \frac{p_n^2}{2m} + \alpha q_n\right). \quad (7)$$

As before, one can introduce a canonical transformation which implements discrete time evolution. Its generator is  $-L(n, n+1)$ . The resulting equations of the transformation are,

$$P_n^{q^0} = P_{n+1}^{q^0} \quad (8)$$

$$q_{n+1}^0 - q_n^0 = N_n \quad (9)$$

$$P_{n+1}^q - P_n^q = -\alpha N_n \quad (10)$$

$$q_{n+1} - q_n = N_n P_{n+1}^q / m \quad (11)$$

$$P_{n+1}^q{}^2 / 2m + P_{n+1}^{q^0} + \alpha q_n = 0. \quad (12)$$

In deriving these equations we have substituted the momenta canonically conjugate to  $p, p^0$  and  $N$ . The last equation would reproduce in the continuum limit the usual constraint of the parameterized particle. The Lagrange multiplier  $N$  is determined by solving the constraint (12),

$$N_n = \frac{m C_{n+1}}{\alpha P_{n+1}^q}, \quad (13)$$

where,

$$C_{n+1} = \frac{P_{n+1}^q{}^2}{2m} + P_{n+1}^{q^0} + \alpha q_{n+1}, \quad (14)$$

is the discretization of the constraint of the continuum theory.

Which can be used to recast the equation of motion as an explicit canonical transformation between  $n + 1 \rightarrow n$  as,

$$q_n = q_{n+1} - \frac{C_{n+1}}{\alpha} \quad (15)$$

$$P_n^q = P_{n+1}^q + \frac{m C_{n+1}}{P_{n+1}^q} \quad (16)$$

$$q_n^0 = q_{n+1}^0 - \frac{m C_{n+1}}{\alpha P_{n+1}^q} \quad (17)$$

$$P_n^{q^0} = P_{n+1}^{q^0}. \quad (18)$$

Notice that the discrete construction does not exist in the limit  $\alpha \rightarrow 0$ . This prevent us from considering the free particle case, as we argued above. The resulting theory has no constraint, and therefore, distinctly from the continuum case, it has four phase-space degrees of freedom (for a discussion of how to reconcile a discrete theory with a given number of degrees of freedom as an approximation to a continuum theory with a different number of degrees of freedom see our recent paper<sup>4</sup>).

We now proceed to quantize the system. We do it by constructing the unitary transformation which reproduces in the Heisenberg language the equations of motion. We consider wavefunctions  $\Psi(q, q_0)$ .

To begin with, we start by writing the following matrix element,

$$< p'_0, p', n + 1 | p_0, q, n > = < p'_0, p', n | U | p_0, q, n > \quad (19)$$

where  $|p_0, q, n\rangle$ ,  $\langle p'_0, p', n|$  are bases of the states at instant  $n$  labeled by the eigenvalues of  $\hat{P}_n^{q^0}$ ,  $\hat{q}_n$ , and  $\hat{P}_n^{q^0}$ ,  $\hat{P}_n^q$ , respectively. Let us now to consider the operator version of equation (12),

$$\begin{aligned} \langle p', p'_0, n+1 | \frac{(\hat{P}_{n+1}^q)^2}{2m} + \hat{P}_{n+1}^{q^0} + \alpha \hat{q}_n | p, q, n \rangle = \\ (\frac{p'^2}{2m} + p'_0 + \alpha q) \langle p', p'_0, n+1 | p, q, n \rangle = 0. \end{aligned} \quad (20)$$

This implies that

$$\begin{aligned} \langle p', p'_0, n+1 | p_0, q, n \rangle = f(p'_0, p_0, q) \delta(p' + \sqrt{2mz}) H(z) \\ + g(p'_0, p_0, q) \delta(p' - \sqrt{2mz}) H(z), \end{aligned} \quad (21)$$

with  $z = -p_0 - \alpha q$ ,  $H(z)$  the step function, and  $f(p', p, q)$  is a normalization factor to be determined by the other set of equations in a similar fashion. Time translation invariance of  $\hat{P}^{q^0}$  is also naturally implemented by replacing (8) in (20) which immediately implies that it is proportional to  $\delta(p'_0 - p_0)$ . On a similar fashion, we implement the remaining equations (9-11) where the Lagrange multiplier has been substituted via equation (13) and a factor ordering has been chosen. The final result for the evolution operator is,

$$\begin{aligned} \langle p', p'_0, n | U | p_0, q, n \rangle = A \frac{1}{z^{1/4}} [\delta(p^0 - p^0) \delta(p' + \sqrt{2mz}) e^{-ipq} \\ - \delta(p^0 - p^0) \delta(p' - \sqrt{2mz}) e^{-ipq}] H(z). \end{aligned} \quad (22)$$

It can be seen that this operator is not strictly speaking unitary, since  $U^\dagger U = 1$  but  $UU^\dagger \neq 1$ . It is what is technically called an “isometry”. A correct quantization can be based on isometries<sup>9</sup>, but we will not discuss this point here. It should also be noticed that if one restricts the motion to certain regions of configuration space before quantizing such that one avoids the inversion point ( $p = 0$ ) in the motion, then the operator is strictly unitary.

Although the quantization is complete, as we stressed before, its physical interpretation is problematic since the “evolution” variable  $n$  does not have any intrinsic meaning. The wavefunctions however, have a correct probabilistic interpretations and therefore can be used to compute the conditional probabilities of the Page–Wootters relational approach.

Let us begin by introducing the conditional probability to obtain  $q = x$  given  $q^0 = t$  (strictly speaking we should phrase the question in terms of

intervals, since the operators have continuous spectra, we omit them here to simplify the notation),

$$P(q = x|q^0 = t) = \frac{\sum_{n=-\infty}^{\infty} |\Psi(x, t, n)|^2}{\sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} dx |\Psi(x, t, n)|^2} \quad (23)$$

Where we have used the standard formula,

$$P_{\text{sim}}(x, t) = P(t)P_{\text{cond}}(x|t) \quad (24)$$

where  $P_{\text{sim}}(x, t) = \lim_{N \rightarrow \infty} \frac{1}{2N} \sum_{n=-N}^N P_{\text{sim}}(x, t)(n)$ , and  $P_{\text{sim}}(x, t)(n)$  is the probability of a *n-simultaneous* measurement of  $q, q^0$ . The sum in  $n$  is due to the lack of knowledge of the actual value of  $n$  when the measurement takes place. Generically, equation (23) may face convergence difficulties. We will see that for the system in question, and for a judicious choice of time, no convergence problems appear.

Notice that the above definition of probability in principle allows other choices of time variable, for instance,

$$P(q^0 = t|q = x) = \frac{\sum_{n=-\infty}^{\infty} |\Psi(x, t, n)|^2}{\sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} dt |\Psi(x, t, n)|^2} \quad (25)$$

which would correspond to the computation of the “time of arrival”, a notoriously subtle problem (see for instance <sup>10</sup> and references therein.) This will require more detailed study, which we will not attempt here, but it is interesting to notice that the expression is well defined.

Let us now explicitly compute, for the system of interest, the expression (23). The calculation is not straightforward unless certain approximations are made. We need to compute the operator  $U$  explicitly (before we had just computed its matrix elements). In order to do this, we write the operator as an exponential of a “Hamiltonian”  $U = e^{-iH}$  and therefore  $\hat{q}_n = e^{-iH} \hat{q}_{n+1} e^{iH}$ . The explicit expression for  $H$  can be worked out as a power series in  $C/P^{q^2}$ . Here we consider an approximation based on the leading term in the series. We will also assume we are close to the continuum limit. The latter occurs when the lapse goes to zero. By studying the solution to the classical equations of motion (8-11) one can see that the continuum limit is achieved in the limit in which  $|P^q| \gg |C|$  (far away from the turning point). In this limit, only the first term of the power series contributes and the explicit expression for  $H$  is,

$$H_{n+1} \sim \frac{mC_{n+1}^2}{2\alpha P_{n+1}^q}. \quad (26)$$

We can therefore quantum mechanically approximate the evolution operator for  $n$  steps  $U(n) \equiv U^n$  by  $e^{-iHn} = \exp\left(-i\frac{mC^2(p_0, p, q)}{2\alpha p}n\right)$ , with  $C = p_0 + h(q, p)$  the constraint, and in our case  $h = \frac{p^2}{2m} + \alpha q$ .

We now would like to study states in which the variable  $q^0$  behaves approximately as a time variable, that is a wave-packet centered around a given value with small dispersion and that does not spread significantly upon evolution with respect to the parameter  $n$ .

We now expand the ‘‘Hamiltonian’’ as  $H = \frac{mp_0^2}{2\alpha p} + \frac{mh^2}{2\alpha p} + \frac{p_0mh}{\alpha p}$ . The first term would produce dispersion in the variable  $q^0$ . The introduction of dispersion by the evolution in  $n$  implies that for large values of  $n$  the system loses correlation between the time variable and  $n$ . In a realistic quantum system, the clock variable will interact with other degrees of freedom inducing decoherence and keeping small the dispersion in the time variable. In our simple model, it is possible to control the dispersion in  $t$  without introducing additional degrees of freedom by restricting the number  $N$  of discrete steps by  $N \ll (\alpha p \sigma_t)/m$ , where  $\sigma_t$  is the initial standard deviation of the time variable. This will allow us to use a sharply peaked wave packet for the time variable in the whole range of values of  $n \in [-N, N]$ . The second term is the final effective Hamiltonian for the  $q$ -part of the system. The third term will generate the evolution for the clock in the regime we are considering. Let us examine more closely the denominator of (23). Noticing that  $U^\dagger(n)|x, t, 0\rangle = |x, t, n\rangle$  and similarly for the bra state, we have that each term in the sum in the denominator is

$$\int dx \langle \Psi | U^\dagger(n) | x, t, 0 \rangle \langle x, t, 0 | U(n) | \Psi \rangle = \int dx \langle \Psi | e^{iHn} | x, t, 0 \rangle \langle x, t, 0 | e^{-iHn} | \Psi \rangle. \quad (27)$$

Now let us assume an initially normalized wave packet of the form  $\Psi(x, t, n = 0) = \phi_0(x)\psi_0(t)$ , which would correspond to a weak coupling between the clock and the particle. Ignoring factor ordering corrections of order  $\frac{1}{p^2}$  in units  $\hbar = c = 1$  we get,

$$\sum_n \int dx \langle \Psi | e^{ip_0 \frac{mh}{\alpha p} n} e^{i \frac{mh^2}{2\alpha p} n} | x, t, 0 \rangle \langle x, t, 0 | e^{-ip_0 \frac{mh}{\alpha p} n} e^{-i \frac{mh^2}{2\alpha p} n} | \Psi \rangle = \sum_n \langle \psi_0 | e^{ip_0 \langle \frac{mh}{\alpha p} \rangle n} | t, 0 \rangle \langle t, 0 | e^{-ip_0 \langle \frac{mh}{\alpha p} \rangle n} | \psi_0 \rangle. \quad (28)$$

To derive this identity, we notice that  $\exp(-i \frac{mh^2}{2\alpha p})$  is an operator that only acts on  $\phi_0(x)$  and therefore can be transferred to the first inner product

where it cancels with  $\exp(i\frac{mh^2}{2\alpha p})$ . In the last step we have neglected fluctuations on the expectation value of  $\frac{mh}{\alpha p}$  replacing it by  $\langle \frac{mh}{\alpha p} \rangle (0)$ , the initial expectation value, when integrating in  $dx$ . Thus, in the end, we get in the denominator the factor:

$$\sum_n |e^{-i\hat{p}_0 \langle \frac{mh}{\alpha p} \rangle n} \psi_0(t)|^2 = \sum_n |\psi_0(t - \langle \frac{mh}{\alpha p} \rangle n)|^2. \quad (29)$$

To appreciate better the meaning of this expression it is best to replace the sum by an integral (since we are in the continuum limit). In the continuum limit, it should hold that  $\langle \frac{mh}{\alpha p} \rangle \ll T$ , that is the elementary step in time should be much smaller than the total period of validity of the relational approximation. Assuming that the state is normalized, the integral yields a factor  $\langle \frac{mh}{\alpha p} \rangle^{-1}$  for the denominator of the conditional probability. Studying now the numerator in a similar fashion we end with a term of the form,

$$\begin{aligned} \sum_n |e^{i\hat{p}_0 \langle \frac{mh}{\alpha p} \rangle n} e^{i\frac{mh^2}{2\alpha p n}} \phi_0(x) \psi_0(t)|^2 = \\ \sum_n |e^{i\frac{mh^2}{2\alpha p} n} \phi_0(x) \psi_0(t - \langle \frac{mh}{\alpha p} \rangle n)|^2, \end{aligned} \quad (30)$$

where we have replaced the interaction term by its average value assuming that  $|\langle p \rangle| \gg \Delta p$ , where  $\Delta p$  is the initial uncertainty of the wave packet in  $p$ . Now we may construct for the initial condition a normalized wave function of the form  $\phi_0(t) = \delta_\epsilon(t)$  peaked around  $t=0$ . Replacing again the sum by an integral we cancel the denominator integrating the delta function after replacing<sup>a</sup>  $n$  by  $\frac{t}{\langle \frac{mh}{\alpha p} \rangle}$ . The final expression turns out to be:

$$P(q = x | q^0 = t) = |e^{i\frac{h^2}{2\alpha p} \times \frac{t}{\langle \frac{h}{\alpha p} \rangle}} \phi_0(x)|^2 \sim |e^{i\frac{h^2}{2p} \times \langle \frac{h}{p} \rangle^{-1} t} \phi_0(x)|^2. \quad (31)$$

Therefore we end with a Schrödinger picture that, as we shall see, reproduces standard quantum mechanics. In order to see that the traditional behavior is recovered, let us consider the Schrödinger picture with

<sup>a</sup>Notice that in general we will not end with an exact Schrödinger description, since the sum in  $n$  may give more than one factor if the dispersive effects and the decoherence by the environment is included. This spreading effect of the clock variable would imply information loss in the *time* evolution of the wave packet. We have neglected this correction here in order to compare with standard quantum mechanics. However, the information loss is a natural consequence of the relational approach which may lead to interesting new physics. We shall show this in a forthcoming paper.

$\hat{h}_{\text{eff}} \equiv \frac{\hat{h}^2}{2\hat{p}} \times \langle \frac{h}{p} \rangle^{-1} (0)$ . Ignoring factor ordering terms, and also second order terms in  $1/p$ , we get, for the evolution of  $\langle q \rangle$  and  $\langle p \rangle$  the equations:

$$\begin{aligned} \frac{d}{dt} \langle q \rangle &= \frac{1}{i} \langle [q, h_{\text{eff}}] \rangle = \langle -i[q, h^2/2p] \rangle \langle \frac{h}{p} \rangle^{-1} (0) = \\ &\langle \frac{h}{p} \rangle^{-1} (0) [\langle (h^2/2p^2) \rangle (t) + \langle h \rangle (t)/m] = \\ \langle h/p \rangle^{-1} (0) \langle h \rangle (t)/m + 1/2 \langle (h^2/p^2) \rangle (t) / \langle h/p \rangle (0), \quad (32) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \langle p \rangle &= \frac{1}{i} \langle [p, h_{\text{eff}}] \rangle = \langle -i[p, h^2/p] \rangle \langle \frac{h}{p} \rangle^{-1} (0) = \\ &-\alpha \langle \frac{h}{p} \rangle^{-1} (0) \langle \frac{h}{p} \rangle (t). \quad (33) \end{aligned}$$

The second equation reproduces the standard result  $\frac{d}{dt} \langle p \rangle = -\alpha$  for  $\frac{\alpha T}{\langle p \rangle (0)} \ll 1$ , which ensures that the expectation value  $\langle h/p \rangle (t)$  does not depart much from its initial value during “evolution”. One can also recover the usual result for the first equation  $\frac{d}{dt} \langle q \rangle = \langle p \rangle (0)/m$  provided  $\langle h \rangle (0) / \langle p \rangle (0) \ll 1$ . This last requirement is a consequence of the two previous conditions on  $\langle p \rangle$ . One can show that it is possible to assign numerical values to the different variables in such a way that the complete set of conditions is satisfied for sufficiently long periods of the  $q^0$  time. One can also check that small uncertainty states can be preserved during this period allowing to replace operators for expectation values.

The correct quantum behavior is therefore extremely limited for this system compared with the traditional Schrödinger quantization using the Newtonian time. This is, however, expected since it is a direct consequence of dealing with time as a quantum operator. Furthermore, for recovering the usual quantum behavior it is necessary to consider the semiclassical properties of the clock including extra degrees of freedom and the effects of the environment. We have shown nonetheless that the relational approach allows us for the first time to describe nature in a completely quantum way. Notice as an interesting consequence of the model that energy, in the form of  $h$ , is not exactly conserved. In fact,  $\frac{\Delta h}{h} \sim \frac{\langle h \rangle T}{\langle h \rangle} \sim \frac{\alpha}{\langle p \rangle (0)} T \ll 1$  and therefore the relative change in energy is small in the range of validity of the approximation.

## 5. Discussion

We have shown that we can, through the use of the consistent discretization approach, formulate a relational quantum mechanics for the parameterized particle. As in the case of general relativity the main ingredient is that the consistent discretization gets rid of the constraint(s) of the system and therefore one can consistently implement the relational description without the usual objections. The example considered is therefore simple yet powerful. To our knowledge there has never been a satisfactory relational quantum description in the past for this system.

The relational description recovers usual quantum mechanics when the discrete approximation approaches the continuum limit and when a clock variable can be isolated that behaves sufficiently close to a classical clock. The relational description does not require any of these assumptions; it exists in much more general situations. In such situations however, there is no sense in seeking a correspondence with usual quantum mechanics.

The example we consider is important, but nevertheless limited. It assumes the existence of a time variable whose simultaneity surfaces are “transverse” to all the classical trajectories of the system. In such situations it is known that the theory can be de-parameterized and therefore one can solve immediately the problem of time (although not through a relational quantum description). Even the same simple example we have analyzed could be shown to exhibit non-transversality by choosing the variable  $q$ . We will analyze this in a future paper. We would also like to analyze a system where the coupling between the clock and the system is less strong. This will require more complexity, introducing extra degrees of freedom. A further generalization we would like to study is systems where the constraint does not have variables that appear linearly —and therefore de-parameterization is challenging—, as the Barbour-Bertotti model. Unfortunately, the latter has a constraint that only depends on the momenta and therefore our discretization technique does not apply. A model where it could be tested is the  $SL(2, R)$  model of Montesinos, Rovelli and Thiemann<sup>11</sup> with two Hamiltonian constraints.

Another problem that has received quite a bit of attention is the fact that it could happen that the system is such that there are an infinite number of crossings of the surface of simultaneity for a given choice of time. Indeed, for our approach this is a problem, since in such a case the denominator of the conditional probability blows up and therefore probabilities vanish. An example of this would be to have a perfect oscillator as a clock.

We see this as an inescapable problem that, however, can be easily circumvented. A “clock” does not have to be associated with a single variable of the problem. In general one will need a collection of variables to have a good clock (just like one needs a wristwatch and a calendar to keep time and date). For instance two oscillators with frequencies whose ratio is not a rational number would suffice.

Another issue that has received some attention, for instance emphasized by Unruh and Wald <sup>12</sup>, is that if one has a time variable that is monotonous implies that the conjugate variable will not be bounded below. Since most Hamiltonians in physics are bounded below, this appears as an obstruction. As we argued above, when we chose  $t = q$ , we do not need a monotonous variable for our definitions to exist (although their interpretation can be delicate). Furthermore, a two-oscillator system with non-rational ratio of frequencies behaves as a perfectly good monotonous clock despite having a Hamiltonian bounded from below. This is due to the fact that we may choose the clock, in principle, as a function of any set of variables whose Poisson algebra with  $h$  would not have reason to be the canonical one. In the example we worked out in detail we did not address this problem since the Hamiltonian we considered is not bounded below.

A concern that might arise in the use of the discrete approach is that there are infinitely many ways of discretizing a continuum theory and therefore the approach has a large degree of ambiguity. We see this as inevitable. An analogy of interest is the case of the Feynman path integral. There, even for simple systems, one defines the integral via a discretization. Different discretizations are known to lead to the same theory, in different factor orderings. Our point of view is therefore that the discretization ambiguities should be treated as factor ordering ambiguities: they must be sorted by comparison with experiment.

Summarizing, the consistent discrete approach to constrained systems, since it produces an approximate theory that is constraint-free, opens the possibility of solving the problem of time through the construction of a relational quantum mechanics. We have shown in detail how it works for a simple example that captures some of the ingredients of the case of general relativity. To our knowledge this is the first time that a correct relational quantum description has been constructed for this system.

## 6. Acknowledgments

We wish to thank Abhay Ashtekar, Julian Barbour, Peter Hajiček, Jim Hartle, Karel Kuchař, Carlo Rovelli, Lee Smolin, Bill Unruh, for discussions and comments on the manuscript.

This work was supported by grants NSF-PHY0090091, funds of the Horace Hearne Jr. Institute for Theoretical Physics, the Fulbright Commission in Montevideo and PEDECIBA (Uruguay).

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